FAST SWITCHING ANALYSIS OF LINEAR SWITCHED SYSTEMS USING EXPONENTIAL SPLITTING

M. PORFIRI†, D. G. ROBERSON‡, AND D. J. STILWELL‡

Abstract. Stability of periodic switched linear systems with fast switching patterns is studied by combining the method of averaging with exponential splitting. This approach yields less conservative bounds on stabilizing fast switching rates than can be obtained with prior approaches. Such bounds can be useful for analysis and design of switching control laws. The method is also generalized to arbitrary (including nonperiodic nonswitched) time-varying systems. In particular, the effects of time-varying state perturbations on the stability of linear periodic switched systems are analyzed.

Key words. averaging method, exponential splitting, fast switching, stability, switched systems, linear systems

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1. Introduction. A switched system is a dynamical system which consists of a family of time-invariant subsystems and a policy that oversees the switching among them. For example,

\[ \dot{x} = A_{\rho(t)} x, \]

where \( \rho(t) \) is a piecewise constant switching signal that selects from a family of matrix-valued coefficients \( \Theta = \{ A_1, A_2, \ldots, A_M \} \), represents such a system. This class of dynamical systems finds extensive application in various areas of engineering practice such as power electronics, network communication, hybrid control, traffic flow, biosystem modeling, etc. (see, e.g., [16] and references therein). Switched systems have been studied for several decades in the systems and control literature (see, e.g., [8]).

Switching among the possible system configurations may be orchestrated according to different protocols, as discussed in [16] and [19]. Switching events may occur as a function of the system state, or at given instants in time independent of the system state. In the latter case, when one or more of the subsystems (the elements of \( \Theta \)) are stable, dwell time approaches, where the switching signal selects stable subsystems for relatively long time intervals, are effective in ensuring system stability. However, when no subsystems are stable, or when system constraints prevent long intervals between switching instants, alternate strategies must be considered. Periodic switching is another well-studied strategy (see, e.g., [20]) that can be useful when such restrictions exist.

Periodic switching is a state-independent switching scheme that utilizes a periodic rule for selecting subsystems. That is, there is a \( T > 0 \) such that \( \rho(t) = \rho(t + T) \) for
all $t \geq 0$. The stability of the resulting periodic linear system may be analyzed using a number of different techniques. The spectral properties and contractibility of the state transition matrix over a switching period may be used to characterize system stability. Such an approach involves computation and multiplication of matrix exponentials, which may present computational challenges (see, e.g., [10]), especially as the system dimension and the number of subsystems grow. Standard Floquet arguments may also be used to study the periodic system, as in [6]. The main drawback of this approach is its inherent dependence on matrix logarithms, which limits qualitative system analysis and control design. In [4] sufficient conditions for uniform asymptotic stability are stated in terms of convex combinations of the matrix measures of the elements of $\Theta$. In [13] the conditions of [4] are generalized by including matrix condition numbers in the sufficient conditions. In [17] it is shown that the stability of a switched system may be assessed by studying an auxiliary time-invariant system whose state coefficient matrix is a certain time average of the original system. In particular, it is shown that the stability of the average system is inherited by the switched system whenever the switching signal is sufficiently fast, or equivalently when the switching period $T$ is sufficiently small (fast switching). In [20] the average system is used to design a stabilizing feedback law for periodic switched systems. More specifically, it is shown that if the average system is stabilizable and detectable, then for any feedback matrix that stabilizes the average system there exists a minimum switching rate that guarantees stability of the periodic system.

Computing the slowest allowable switching rate (or the maximum switching period) is an important factor to consider when analyzing stability and designing feedback control laws. In [17] the estimate of the maximum switching period is based on a perturbation analysis of the system spectral properties. The estimate may also be established by specializing to switched linear systems the method of averaging proposed in [7] for general linear time-varying systems (see, e.g., [11]). The approach of [7] relies on a decomposition of the system transition matrix into a part due to the time average of the system dynamics and a perturbation that is on the order of $T^2$, where $T$ is the switching period. For the time-varying system $\dot{x} = A(t)x$, the state transition matrix satisfies

$$
\Phi(s + \tau, s) = e^{\tilde{A}_\tau(s)\tau} + E_\tau(s),
$$

where

$$
\tilde{A}_\tau(s) = \frac{1}{\tau} \int_s^{s+\tau} A(\sigma)d\sigma
$$

and

$$
\|E_\tau(s)\| \leq \alpha^2 \tau^2 e^{\alpha\tau}
$$

if $\|A(t)\| \leq \alpha$ for all $t$. This decomposition is well known and appears in several textbooks (see, e.g., [1] and [12, Exercise 4.25]). Henceforth, this approach to estimating the maximum switching period is referred to as the standard approach, and the estimate itself as the standard estimate. The estimate of [17] is difficult to compute numerically, as it involves quantifying the sensitivity of matrices to perturbations. The standard estimate is easily applicable and requires the knowledge of elementary quantities. Nevertheless, it generally leads to conservative results that may be even a few orders of magnitude less than the exact value computed using Floquet theory.
The present work is focused on characterizing the slowest stabilizing switching rate through a computationally tractable expression. Loosely speaking, improved bounds are due to directly expressing the effects of the commutation relationship of the elements of Θ. The general idea stems from combining the standard approach with well-developed results from numerical algebra (see, e.g., [14, 10]). By exploiting the concept of matrix exponential splitting, which is widely used in numerical analysis of partial differential equations (see, e.g., [9]), a novel switching rate estimate is obtained. The estimate bounds the slowest switching rate that guarantees uniform asymptotic stability of switched systems characterized by asymptotically stable averages. The new estimate is compared to the standard estimate and its improvement is shown. These results are also extended to a more general linear class of dynamical systems, and new results on the method of averaging are established. In particular, the effects of time-varying state perturbations on the stability of linear periodic switched systems are analyzed.

The paper is organized as follows. Section 2 contains the general system description and a discussion of fundamental matrix measure and exponential splitting concepts. Section 3 addresses our principle contribution, an improved estimate for the maximum switching period for stability under fast switching. Section 4 addresses the stability analysis of more general dynamical systems and presents stability conditions for perturbed switched systems. Section 5 contains concluding remarks.

2. System description. We consider the homogeneous linear time-varying system

\[ \dot{x} = A(t)x, \]  

where \( t \in \mathbb{R}^+ \) indicates the time variable, \( x \in \mathbb{R}^n \) is the system state, and \( A(\cdot) : \mathbb{R}^+ \to \mathbb{R}^{n \times n} \) is a bounded and right continuous matrix function. By a right continuous function in \( \mathbb{R}^+ \) we mean a function whose value at each point \( t \) in \( \mathbb{R}^+ \) equals its right-hand limit at \( t \). We further assume that the number of discontinuities of \( A \) is finite over any finite interval \([s, s + \tau]\), where \( s, \tau \in \mathbb{R}^+ \). Thus we do not consider systems which exhibit Zeno behavior or chattering, as described in [8] and [16].

The homogeneous systems that we consider include switched systems for which \( A(t) = A_{\rho(t)} \) with \( \rho(t) \) being a switching signal that selects from among a family of constant matrix functions. In section 3 this class of switched systems is considered, with the additional conditions that \( \rho(t) \) is periodic and selects from among a finite set of matrix functions \( \Theta = \{A_1, A_2, \ldots, A_M\} \). The switched systems considered in section 4 are not subject to the additional conditions, so that \( \rho(t) \) may be nonperiodic and \( \Theta \) may be countably infinite. In section 4 we also consider more general linear time-varying systems, where \( A(t) \) may be nonconstant between points of discontinuity. For existence and uniqueness of these types of problems one may refer to [3] and [18].

2.1. Matrix measure. Let \( \| \cdot \| \) denote a norm in \( \mathbb{R}^n \) and the corresponding induced norm in \( \mathbb{R}^{n \times n} \). The one-sided directional derivative of \( \| \cdot \| \) at the identity matrix \( I \in \mathbb{R}^{n \times n} \) in the direction \( A \in \mathbb{R}^{n \times n} \) is called the matrix measure of \( A \) and is denoted by \( \mu(A) \). That is,

\[ \mu(A) = \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h}. \]
Basic properties of the matrix measure for $A, B \in \mathbb{R}^{n \times n}$ and $c \in \mathbb{R}$ are

\begin{align}
(2.3a) \quad & \mu(\mathbf{I}) = 1, \quad \mu(-\mathbf{I}) = -1, \quad \mu(0) = 0, \\
(2.3b) \quad & \mu(cA) = |c|\mu(\text{sgn}(c)A), \quad \mu(A + c\mathbf{I}) = \mu(A) + c, \\
(2.3c) \quad & \mu(A + B) \leq \mu(A) + \mu(B), \\
(2.3d) \quad & \|e^A\| \leq e^{\mu(A)}, \\
(2.3e) \quad & \max_{i=1,...,n} \left| \Re(\lambda_i(A)) \right| \leq \mu(A) \leq \|A\|,
\end{align}

where $\lambda_i(A)$ indicates the $i$th eigenvalue of $A$. These properties hold for the matrix measure defined in any induced matrix norm. Derivation of the relationships (2.3) may be found in [15]. An additional property of the matrix measure (see, e.g., [7]) places upper and lower bounds on the transition matrix norm of the system (2.1) according to the relationship

\begin{equation}
(2.4) \quad e^{-\int_t^\tau \mu(-A(\theta))d\theta} \leq \|\Phi(t, \tau)\| \leq e^{\int_t^\tau \mu(A(\theta))d\theta}
\end{equation}

for all $t, \tau \in \mathbb{R}^+$ and $t \geq \tau$.

The computation of the matrix measure is generally very involved, but there are special cases where reduced effort is needed (see, e.g., [15, 2]). In what follows, we primarily make use of the vector $P$-norm and the corresponding matrix induced norm and matrix measure, defined as

\begin{align}
(2.5) \quad & \|x\|_P = \sqrt{x^T P x}, \quad \|A\|_P = \max_{i=1,...,n} \sigma_i(P^{1/2}AP^{-1/2}), \\
& \mu_P(A) = \frac{1}{2} \max_{i=1,...,n} \lambda_i(P^{1/2}AP^{-1/2} + P^{-1/2}A^TP^{1/2}),
\end{align}

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $\sigma_i(A)$ indicates the $i$th singular value of $A$, and $P$ is any symmetric positive definite matrix in $\mathbb{R}^{n \times n}$. In particular, when $P = I$ the matrix measure corresponding to the Euclidean norm is obtained.

### 2.2. Exponential splitting

Exponential splitting methods are numerical tools for computing matrix exponentials (see, e.g., [9]). They involve decomposing a matrix into the sum of matrices, whose exponentials can be easily computed, and approximating the original matrix exponential in terms of the exponentials of the simpler matrices. That is, traditional splitting methods approximate the matrix exponential $e^{(A+B)h}$ by the product $e^{Ah}e^{Bh}$, where $A, B \in \mathbb{R}^{n \times n}$ and $h \in \mathbb{R}^+$, and where $e^{Ah}$ and $e^{Bh}$ are relatively easy to compute.

We apply splitting methods in the opposite direction. That is, we approximate the product of exponentials $e^{Ah}e^{Bh}$ by the single exponential $e^{(A+B)h}$. The motivation for this approach is the recognition that the product represents the state transition matrix for a switched system, and the single exponential represents the transition matrix for a corresponding average system. We consider the problem of estimating the global error in the first order exponential splitting by defining

$$
E(h) = e^{Ah}e^{Bh} - e^{(A+B)h}.
$$

**Proposition 2.1.** Given $A, B \in \mathbb{R}^{n \times n}$ and $h \in \mathbb{R}^+$,

$$
\|E(h)\| \leq \frac{1}{2} h^2 \|A, B\|_P e^{h(\mu(A)+\mu(B))},
$$
where
\[
[A, B] = AB - BA
\]

is the commutator of \( A \) and \( B \).

Proof. Part of the following arguments are borrowed from [14]. The main difference in those results lies in the estimate itself, where we make use of additional properties of the matrix measure. By differentiating \( E(h) \), we obtain
\[
\frac{d}{dh} E(h) = (A + B)E(h) + [e^{Ah}, B] e^{Bh}.
\]

The above ordinary differential equation (ODE) with homogeneous initial conditions may be solved to obtain
\[
E(h) = \int_0^h e^{(h-\vartheta)(A+B)} S(\vartheta) e^{\vartheta B} d\vartheta,
\]
where
\[
S(\vartheta) = [e^{A\vartheta}, B].
\]

It is clear that \( S(0) = 0 \). By differentiating \( S(\vartheta) \), we obtain
\[
\frac{d}{d\vartheta} S(\vartheta) = AS(\vartheta) + [A, B] e^{A\vartheta}.
\]

The above ODE may be solved, yielding
\[
S(\vartheta) = \int_0^\vartheta e^{(\vartheta-p)A} [A, B] e^{pA} dp.
\]

Substituting (2.7) into (2.6), we obtain
\[
E(h) = \int_0^h e^{(h-\vartheta)(A+B)} \left( \int_0^\vartheta e^{(\vartheta-p)A} [A, B] e^{pA} dp \right) e^{\vartheta B} d\vartheta.
\]

Applying standard matrix norm inequalities, then using property (2.3d) followed by (2.3b), we obtain
\[
\|E(h)\| \leq \int_0^h \left\| e^{(h-\vartheta)(A+B)} \right\| \int_0^\vartheta \left\| e^{(\vartheta-p)A} \right\| \| [A, B] \| \left\| e^{pA} \right\| dp \left\| e^{\vartheta B} \right\| d\vartheta
\]
\[
\leq \| [A, B] \| \int_0^h \mu((h-\vartheta)A) + \mu((h-\vartheta)B) \int_0^\vartheta \mu((\vartheta-p)A) e^{\mu(pA)} dp d\vartheta
\]
\[
= \| [A, B] \| \int_0^h e^{h\mu(A)} e^{h\mu(B)} \int_0^\vartheta \mu((\vartheta-p)A) e^{\mu(A)} dp d\vartheta
\]
\[
= \| [A, B] \| \int_0^h e^{h\mu(A)} e^{h\mu(B)} \int_0^\vartheta dp d\vartheta
\]
\[
= \frac{1}{2} h^2 \| [A, B] \| e^{h(\mu(A) + \mu(B))}. \quad \square
\]

The problem of estimating the global error in the first order exponential splitting has also been studied in [5], using a Lie algebraic framework. The resulting error bound in Proposition 2 of [5] involves the determination of the limit of a series of nested commutators, arising from the Baker–Campbell–Hausdorff formula, which can be potentially difficult to compute.
3. Stability analysis of periodic switching systems. The present section is devoted to the analysis of homogeneous time-varying systems of the form (2.1), where \( A(t) \) is a \( T \)-periodic piecewise constant bounded matrix function. At any point in time, \( A(t) \) takes values from the set \( \Theta = \{ A_1, A_2, \ldots, A_M \} \) according to the policy

\[
A(t) = A_m \quad \text{for} \quad t \in \left[ kT + T \sum_{i=0}^{m-1} \delta_i, kT + T \sum_{i=0}^{m} \delta_i \right),
\]

where \( \delta_i \) is the duty cycle of the subsystem corresponding to the matrix \( A_i \) with the convention \( \delta_0 = 0 \), so that \( 0 \leq \delta_i < 1 \) and \( \sum_{i=1}^{M} \delta_i = 1 \), and where \( k \in \mathbb{N} \). Our goal is to determine an estimate for the maximum switching period \( T \) for which (2.1) is uniformly asymptotically stable.

For the periodic switched linear system (3.1), the transition matrix \( \Phi \) over any period \([kT, (k+1)T)\) is called a monodromy. Due to periodicity, the monodromy over the interval for any \( k \) equals the monodromy over the interval for \( k = 0 \), so that

\[
\Phi(\delta_1 + \delta_2, T\delta_1) \Phi(\delta_2, 0) = e^{T A_M \delta_M} \ldots e^{T A_2 \delta_2} e^{T A_1 \delta_1}.
\]

In what follows we show a sufficient condition for uniform asymptotic stability based on the transition matrix contractibility.

**Proposition 3.1.** If for some norm

\[
\sum_{i=1}^{M} \delta_i \mu(A_i) < 0,
\]

then the system is uniformly asymptotically stable for every positive \( T \). Otherwise, if the average system matrix

\[
\bar{A}_T = \sum_{i=1}^{M} \delta_i A_i
\]

is Hurwitz, then there exists a \( T^* > 0 \) such that when \( T < T^* \) the system is uniformly asymptotically stable. \( T^* \) satisfies

\[
e^{T \mu_P(\bar{A}_T)} + \frac{1}{2}(T^*)^2 \Gamma_P(A) \exp \left( T^* \sum_{i=1}^{M} \delta_i \mu_P(A_i) \right) = 1,
\]

where

\[
\Gamma_P(A) = \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \delta_i \delta_j \| [A_i, A_j] \|_P.
\]

The norm is constructed as in (2.5), with \( P \) being the unique symmetric positive definite solution of

\[
\bar{A}_T^T P + P \bar{A}_T + R = 0,
\]

where \( R = R^T > 0 \) is arbitrary.
The proof of Proposition 3.1 is given at the end of the section, after a preliminary proposition on the application of exponential splitting to the monodromy matrix. We note that although Proposition 3.1 provides sufficient stability conditions, the condition of Hurwitz $\bar{A}_T$ is actually necessary for stability for small $T$. Indeed, if $\bar{A}_T$ is not Hurwitz, there exists a $T^* > 0$ such that the system is unstable for all $T < T^*$, as Theorem 2.1 in [7] indicates. Constructing the norm as in (3.7) guarantees that the measure of $\bar{A}_T$ is negative, given (2.5) and the fact that $\bar{A}_T$ is Hurwitz. If an arbitrary norm is chosen, the corresponding measure of $\bar{A}_T$ may be positive, even if $\bar{A}_T$ is Hurwitz. The commutators appearing in (3.6) reflect the similarity of the eigenvectors of the subsystem state coefficient matrices, a set of relationships not considered in the standard approach. If all of the matrices $A_1, \ldots, A_M$ commute and $\bar{A}_T$ is Hurwitz, the switching system is uniformly asymptotically stable. Indeed, the commutator in (3.5) vanishes for every $i, j$, and the monodromy matrix is a contraction for any $T$. If all of the matrices $A_1, \ldots, A_M$ are symmetric and negative definite, then $\bar{A}_T$ is Hurwitz and the switched system is uniformly asymptotically stable. Indeed, the matrix measure induced by the Euclidean norm satisfies (3.3).

Proposition 3.2. For the switched periodic linear system defined by (3.1), the difference $E_T$ between the monodromy matrix over the period $[0, T)$ and the transition matrix of the average system over the same period, defined by

\[ \Phi(T, 0) = E_T + e^{T\bar{A}_T}, \]

satisfies the norm bound

\[ \|E_T\| \leq \frac{1}{2} T^2 \Gamma(A) \exp \left( T \sum_{i=1}^{M} \delta_i \mu(A_i) \right), \]

where

\[ \Gamma(A) = \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \delta_i \delta_j \|[A_i, A_j]\|. \]

Proof. We rewrite the global error in (3.8) as

\[
E_T = \sum_{i=0}^{M-2} e^{TA_{M+1} \delta_{M+1}} e^{TA_{M} \delta_{M}} \cdots e^{TA_{M+1-i} \delta_{M+1-i}} \\
\times \left( e^{T \delta_{M-i} A_{M-i}} e^{T \sum_{j=i+1}^{M-1} \delta_{M-j} A_{M-j}} - e^{T \sum_{j=i}^{M-1} \delta_{M-j} A_{M-j}} \right),
\]

where $A_{M+1} = I$ and $\delta_{M+1} = 0$. Applying Proposition 2.1 for each bracketed term, the Schwarz inequality, the triangle inequality, and the measure properties (2.3b),
(2.3c), and (2.3d), we obtain

\[ \| E_T \| \leq \sum_{i=0}^{M-2} \left\| e^{T A_{M-i} \delta_{M-i+1}} \right\| \ldots \left\| e^{T A_{M+1-i} \delta_{M+1-i}} \right\| \times \left\| e^{T A_{M-i} T \sum_{j=i+1}^{M-1} \delta_{M-j} A_{M-j}} e^{T \sum_{j=i+1}^{M-1} \delta_{M-j} A_{M-j}} \right\| \]

\[ \leq \frac{1}{2} T^2 \sum_{i=0}^{M-2} \sum_{j=i+1}^{M-1} \delta_{M-i+1} \delta_{M-j} \left\| [A_{M-i}, A_{M-j}] \right\| e^{T \sum_{i=0}^{M-1} \delta_{M-i} \mu(A_{M-i})}. \quad \square \]

**Proof of Proposition 3.1.** To demonstrate uniform asymptotic stability, we show that the monodromy matrix \( \Phi(T, 0) \) is a contraction. This approach relies on the boundedness of the transition matrix over any interval whose duration is less than one switching period, due to the fact that the set \( \Theta \) of state coefficient matrices contains a finite number of elements. If the monodromy is a contraction, \( \| \Phi(t, 0) \| \) can be made arbitrarily small by choosing \( t \) sufficiently large, establishing uniform asymptotic stability (see, e.g., [12, Exercise 6.15]).

To establish the first sufficient condition, we use (3.2) to obtain an expression for the monodromy matrix norm in the norm for which (3.3) holds. Applying the Schwarz inequality and matrix measure properties (2.3b) and (2.3d) yields

\[ \| \Phi(T, 0) \| \leq e^{T \delta_{M} \mu(A_{M})} \ldots e^{T \delta_{2} \mu(A_{2})} e^{T \delta_{1} \mu(A_{1})} = e^{T \sum_{i=1}^{M} \delta_{i} \mu(A_{i})} < 1. \]

This condition appeared in [4] for the Euclidean norm.

To establish the second sufficient condition, we use (3.8) to obtain a matrix measure norm expression. Using the triangle inequality together with (2.3d) yields

\[ \| \Phi(T, 0) \| \leq \| E_T \| + e^{T \mu(\tilde{A}_T)}. \]

Thus, considering Proposition 3.2, we look for switching periods satisfying

\[ \frac{1}{2} T^2 \Gamma(A) \exp \left( T \sum_{i=1}^{M} \delta_{i} \mu(A_{i}) \right) + e^{T \mu(\tilde{A}_T)} < 1. \]

We specialize to the norm defined in (2.5), where \( P \) is the solution of (3.7). Because the matrix \( P \)-norm is induced from the vector \( P \)-norm, properties (2.3) hold for the matrix measure derived from this norm. The measure of the average matrix is

\[ \mu_P(\tilde{A}_T) = \frac{1}{2} \max_{i=1, \ldots, n} \lambda_i \left( P^{1/2} \tilde{A}_T P^{-1/2} + P^{-1/2} \tilde{A}_T^T P^{1/2} \right) \]

\[ = \frac{1}{2} \max_{i=1, \ldots, n} \lambda_i \left( P^{-1/2} P \tilde{A}_T P + P P \tilde{A}_T P \right) \]

\[ = \frac{1}{2} \max_{i=1, \ldots, n} \lambda_i \left( P^{-1/2} R P^{-1/2} \right). \]

Because both \( P \) and \( R \) are positive definite, it follows that \( \mu_P(\tilde{A}_T) < 0 \). For the chosen norm, then, the second term of the LHS of (3.11) is a decreasing function of
which attains the value 1 at $T = 0$. On the other hand, the first term is a function that increases with $T$, since (3.3) does not hold by hypothesis. However, the first term vanishes at $T = 0$, and, moreover, its derivative with respect to $T$ is 0 at $T = 0$. Observing that the LHS of (3.11) is continuous in $T$, it can be characterized as a function that equals 1 at $T = 0$, decreases for small $T$, and increases with larger $T$. Hence there is a time $T^*$ such that (3.5) holds. Because the LHS of (3.11) provides an upper bound on $\|\Phi(T, 0)\|$, it follows that when $T < T^*$, the monodromy matrix is a contraction with respect to $\| \cdot \|_p$. \[\square\]

In [13], condition (3.3) for the Euclidean norm is generalized by applying a modified version of property (2.3d), where a similarity transformation is exploited and the norm of the matrix exponential is bounded by using the condition number of the transformation and the measure of the transformed matrix. In [13], it is shown that this condition may yield equivalent, less conservative, or more conservative results than (3.3), depending on the problem at hand and on the used similarity transformation.

Since they are defined in terms of a matrix norm, the matrix measure and commutator norm quantities appearing in (3.5) and (3.10) do not grow with matrix dimension. Therefore, the estimate of the slowest stabilizing switching period provided by (3.5) does not degrade with system dimension. It should be noted that because (3.10) accounts for all possible commutation relationships between subsystem matrices, it tends to make the estimate of (3.5) more conservative as the number of subsystems grows. Nevertheless, it is always less conservative than the standard estimate represented by the RHS of (3.12). The following proposition formalizes this result.

**Proposition 3.3.** For the switched system in (3.1), it is the case that

\[
\frac{1}{2} T^2 \Gamma(A) \exp \left( T \sum_{i=1}^{M} \delta_i \mu(A_i) \right) \leq T^2 \alpha^2 e^{T\alpha},
\]

where $\Gamma(A)$ is defined in (3.10) and

\[\alpha = \text{esssup}_{t \in [0, T]} \|A(t)\| = \max_{i=1, \ldots, M} \|A_i\|.
\]

**Proof.** By noticing that for any $i, j \in \{1, \ldots, M\}$,

\[\| [A_i, A_j] \| \leq 2 \| A_i A_j \| \leq 2 \alpha^2,
\]

and that

\[\sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \delta_i \delta_j \leq 1,
\]

a trivial application of the matrix measure properties (2.3c) and (2.3e) leads to the claim (3.12). \[\square\]

The standard estimate shows severe limitations since it discards the properties of the commutators, and, moreover, never yields decreasing global estimates. Loosely speaking, the standard estimate ignores the relations between the eigenvectors of the state matrices that are accounted for by the commutator. This means that even when (3.3) is satisfied, or when the matrices all commute and have a Hurwitz average, it still provides an upper bound for the maximum switching period.

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As a numerical example, consider the two-dimensional periodic switched system from [19], whose state matrices are
\[ A_1 = \begin{bmatrix} -4 & -5 \\ 7 & 7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & 5 \\ -7 & -13 \end{bmatrix}, \]
with the normalized time durations \( \delta_1 = 0.6, \delta_2 = 0.4 \). It is easy to check that (i) \( A_1 \) and \( A_2 \) are not Hurwitz; (ii) \( \bar{A}_T \) is Hurwitz; and (iii) condition (3.3) for Euclidean norms is not satisfied. We estimate the maximum time that guarantees the stability of the switched system with three different methods: (i) solving (3.5) as elucidated by Proposition 3.1, with the matrix measure defined by the norm computed with \( R = I \); (ii) solving (3.5) upon using the error bound (3.12) as done in [11]; and (iii) determining directly the maximum switching period that guarantees the contraction property of the monodromy matrix with respect to the norm suggested in Proposition 3.1. These estimates are compared with the exact solution obtained by requiring that the characteristic multipliers (see, e.g., [12]) be less than one. In Table 1, we report the results of the computations. We emphasize that both the standard and the present approaches approximate the estimate obtained from the contraction condition. Nevertheless, the present approach provides a better estimation, as predicted by the theoretical analysis and illustrated by the numerical example.

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Although the expression (3.5) appears complicated, it actually provides a convenient computational approach to estimating the critical switching period \( T^* \). Indeed, \( \Gamma(A) \) defined in (3.10) is independent of \( T^* \), as are the matrix measures of the individual subsystems and the average system. In contrast, the exact solution from Floquet theory requires expensive matrix logarithm computations that must be repeated for each candidate \( T^* \). Any algorithm using Floquet theory should consider small time steps beginning at zero to account for the oscillatory behavior of Floquet exponents.

**4. Stability analysis of general linear time-varying systems.** In the present section we focus on the generalization of the method of averaging to arbitrary time-varying linear systems. The state matrix in (2.1) may be continuous, in which case the theory developed in this section provides a useful bound on the error incurred by estimating the transition matrix of the time-varying system over some time interval by the transition matrix of the average system over the same time interval. The state matrix may have points of discontinuity, however. In contrast to the switching systems studied in section 3, the state matrix \( A(t) \) may vary between the discontinuities, and the system is not constrained to be periodic.

We begin by specifying a technique for estimating the transition matrix of (2.1) over an interval. The method involves dividing the interval into a large number of subintervals and approximating the state matrix over each subinterval by its value at the beginning of the subinterval. The transition matrix of the time-varying system (2.1) is then approximated by the transition matrix of the resulting switched system.
Proposition 4.1. Suppose that $A(t)$ in (2.1) is right continuous, with a finite number $D$ of discontinuities on the interval $[s, s + \tau]$ for any $s, \tau \in \mathbb{R}^+$. Consider a set of $N + 1$ time instants $\{t_i\}$ in $[s, s + \tau]$, where $s = t_0 < t_1 < \cdots < t_N = s + \tau$, consisting of the union of the points of discontinuity of $A(t)$ and the points $s + k\frac{\tau}{M}$, where $k = 0, \ldots, M$, so that $M \leq N \leq M + D$. Define $A_i^\# = A(t_{i-1})$ for all $i = 1, \ldots, N$. Let $A^\#(t)$ be the corresponding switched system state matrix, whose pointwise in time value is selected from the set $\Theta^\# = \{A_1^\#, \ldots, A_N^\#\}$. Let $\Phi(t, s)$ be the transition matrix associated with $A(t)$, and let $\Phi^\#(t, s)$ be the transition matrix associated with $A^\#(t)$. Then $\Phi^\#(t, s) \to \Phi(t, s)$ uniformly on $[s, s + \tau]$ as $M \to \infty$, that is, as $N \to \infty$.

Proof. Let $R(t, s) = \Phi^\#(t, s) - \Phi(t, s)$, and differentiate to obtain
\begin{equation}
R(t, s) = A^\#(t)\Phi^\#(t, s) - A(t)\Phi(t, s) = A(t)R(t, s) + (A^\#(t) - A(t))\Phi^\#(t, s).
\end{equation}
Since $R(t, s) = 0$ at $t = s$, the solution to the differential equation (4.1) is
\begin{equation}
R(t, s) = \int_s^t (A^\#(\xi) - A(\xi))\Phi^\#(\xi, s)d\xi.
\end{equation}
The error may be bounded as
\begin{equation}
\|R(t, s)\| \leq \int_s^t \|\Phi(\xi, s)\| \|A^\#(\xi) - A(\xi)\| \|\Phi^\#(\xi, s)\| d\xi \leq \alpha \beta \gamma \tau,
\end{equation}
where
\begin{align*}
\alpha &= \operatorname{esssup}_{s, t \in [s, s + \tau]} \|\Phi(t, s)\|, \quad \beta = \operatorname{esssup}_{t \in [s, s + \tau]} \|A^\#(t) - A(t)\|, \quad \gamma = \operatorname{esssup}_{s, t \in [s, s + \tau]} \|\Phi^\#(t, s)\|.
\end{align*}
Let
\begin{equation}
\zeta = \operatorname{esssup}_{t \in [s, s + \tau]} e^{\|A(t)\|\tau}.
\end{equation}
Because $A(t)$ has a finite number of discontinuities, $\|A(t)\|$ is bounded on the compact interval $[s, s + \tau]$. Using (2.3d), (2.3e), and (2.4), it follows that both $\alpha$ and $\gamma$ are bounded by $\zeta$, a quantity that is not influenced by switching behavior. As $N \to \infty$, the maximum time interval between switching instants vanishes, and right continuity of $A(t)$ ensures that $\beta \to 0$. It follows that $\|R(t, s)\| \to 0$, so that the error between the transitions matrices $\Phi^\#(t, s)$ and $\Phi(t, s)$, at every point $t \in [s, s + \tau]$, is bounded by a common vanishing quantity. Therefore $\Phi^\#(t, s) \to \Phi(t, s)$ uniformly on $[s, s + \tau]$ as $N \to \infty$. \qed

This result is now used to bound the difference between the time-varying system transition matrix and the corresponding average system transition matrix over a time interval.

Proposition 4.2. Suppose that $A(t)$ in (2.1) is right continuous and bounded, with a finite number of discontinuities on the interval $[s, s + \tau]$ for any $s, \tau \in \mathbb{R}^+$. The transition matrix $\Phi(s + \tau, s)$ of (2.1) over the interval $[s + \tau, s]$ is given by
\begin{equation}
\Phi(s + \tau, s) = \Phi_\tau(s + \tau, s) + E_\tau(s),
\end{equation}
where $\Phi_\tau(s + \tau, s)$ is the transition matrix of the time-invariant average system represented by the sample average of $A(t)$ on the interval $[s, s + \tau]$. That is, the transition
matrix of
\begin{equation}
\bar{A}_r(s) = \frac{1}{\tau} \int_s^{s+\tau} A(\vartheta) d\vartheta
\end{equation}
and \(E_r(s)\) satisfies
\begin{equation}
\|E_r(s)\| \leq \frac{1}{2} \int_s^{s+\tau} \int_0^{s+\tau} \| [A(\vartheta), A(\xi)] \| d\xi d\vartheta \exp \left( \int_s^{s+\tau} \mu(A(\xi)) d\xi \right).
\end{equation}

Proof. Define the \(N+1\) time instants \(\{t_i\}\) in \([s, s+\tau]\), the switching system state matrix \(A^\#(t)\), and the transition matrix \(\Phi^\#(t, s)\) as in Proposition 4.1. Define the average \(\bar{A}^\#(s)\) of the \(A_i^\#\) matrices over the interval \([s, s+\tau]\) as
\[
\bar{A}^\#(s) = \sum_{i=1}^{N} \delta_i A_i^\#,
\]
where \(\delta_i = \frac{t_i - t_{i-1}}{\tau}\).

Define \(\Phi^\#(s+\tau, s)\) as the transition matrix over the interval \([s, s+\tau]\) of the time-invariant system whose state matrix is \(\bar{A}^\#(s)\), and define the error \(E_r^\#(s) = \Phi^\#(s+\tau, s) - \Phi_r^\#(s+\tau, s)\). From (3.9), \(E_r^\#(s)\) is bounded by
\begin{equation}
\|E_r^\#(s)\| \leq \frac{1}{2} \tau^2 \Gamma(A^\#) \exp \left( \tau \sum_{i=1}^{N} \delta_i \mu(A_i^\#) \right),
\end{equation}
where \(\Gamma\) is defined as in (3.10). According to Proposition 4.1, \(\Phi^\#(s+\tau, s) \to \Phi(s+\tau, s)\) as \(N \to \infty\). Similarly, \(\Phi_r^\#(s+\tau, s) \to \Phi_r(s+\tau, s)\) as \(N \to \infty\). It follows that \(E_r^\#(s) \to E_r(s)\) as \(N \to \infty\). The summations on the RHS of (4.5), including the double summation appearing in \(\Gamma(A^\#)\), are performed over the set of time instants \(\{t_i\}\) on the interval \([s, s+\tau]\), with the size of the steps between time instants determined by the \(\delta_i\). Because of the way the \(N+1\) time instants are defined, the maximum step size approaches zero as \(N \to \infty\). Thus the summations are consistent with the definition of Riemann sums (see, e.g., [3]), and they converge to integrals as \(N \to \infty\). It follows that the RHS of (4.5) converges to the RHS of (4.4), and therefore that condition (4.4) holds.

When the matrix \(A\) is constant, the remainder in (4.2) vanishes. In this case, the present estimate yields the exact result since the commutator is zero. This is in sharp contrast with the standard estimate, which disregards the influence of the commutator, resulting in a generally looser bound depending only on the supremum of the time-varying matrix norm.

As a sample application of Proposition 4.2 (in the following proposition) we consider perturbed periodic switched systems.

**Proposition 4.3.** Consider the system (2.1) with
\begin{equation}
A(t) = F(t) + \varepsilon B(t),
\end{equation}
where \(F\) is a \(T\)-periodic switching matrix of the form (3.1), \(B\) is a bounded right continuous matrix function which has at most a finite number of discontinuities within any switching period, and \(\varepsilon\) scales the perturbation magnitude. Assume that \(B\) has zero average over any switching period. If for some norm
\[
\sum_{i=1}^{M} \delta_i \mu(F_i) + \varepsilon \text{esssup}_{t \in \mathbb{R}^+} \mu(B(t)) < 0,
\]
then the system (4.6) is uniformly asymptotically stable for every positive $T$. Otherwise, if the average matrix
\[
\bar{A} = \sum_{i=1}^{M} \delta_i F_i
\]
of (4.6) is Hurwitz, then there exists a $T^* > 0$ such that when $T < T^*$ the system is uniformly asymptotically stable. $T^*$ satisfies
\[
e^{T^* \mu_P(\bar{A})} + \frac{1}{2} (T^*)^2 (\Gamma_P(F) + 2\beta_1 \epsilon + \beta_2 \epsilon^2) \exp \left( T^* \left( \sum_{i=1}^{M} \delta_i \mu_P(F_i) + \epsilon \beta_3 \right) \right) = 1,
\]
where $\Gamma_P$ is defined as in (3.6), the norm is constructed according to (3.7), and
\[
\beta_1 = \text{esssup}_{t \in \mathbb{R}^+} \| [F_i, B(t)] \|_P, \quad \beta_2 = \text{esssup}_{t, \xi \in \mathbb{R}^+} \| [B(t), B(\xi)] \|_P,
\]
\[
\beta_3 = \text{esssup}_{t \in \mathbb{R}^+} \mu_P(B(t)).
\]

**Proof.** We look for switching periods where the transition matrix of the system (4.6) from $kT$ to $(k+1)T$ is a contraction for every $k \in \mathbb{Z}^+$. To establish the first sufficient condition, we use matrix measure properties (2.4), (2.3c), and (2.3b) to obtain a bound on the transition matrix between consecutive switching events
\[
\| \Phi(kT + i \delta_j T, kT + i \sum_{j=0}^{i-1} \delta_j T) \| \leq \exp \left( \delta_i T \text{esssup}_{t \in \mathbb{R}^+} \mu(\bar{A}(t)) \right)
\]
\[
\leq \exp \left( \delta_i T \mu(F_i) + \delta_i T \epsilon \text{esssup}_{t \in \mathbb{R}^+} \mu(B(t)) \right),
\]
where $\delta_0 = 0$ by convention. Using the Schwarz inequality then yields
\[
\| \Phi((k+1)T, kT) \| \leq \exp \left( T \sum_{i=1}^{M} \delta_i \mu(F_i) + T \epsilon \text{esssup}_{t \in \mathbb{R}^+} \mu(B(t)) \right) < 1.
\]

To establish the second sufficient condition, note that because $B$ has zero average over any switching period, Proposition 4.2 can be applied to the transition matrix of (4.6) to yield
\[
\Phi((k+1)T, kT) = \exp \left( \bar{A}_T T \right) + E_T(kT),
\]
where $\| E_T(kT) \|$ is bounded according to (4.4). We specialize to the norm constructed in (2.5), where $P$ is the unique symmetric positive definite solution of (3.7), and where $R = R^T > 0$ is arbitrary. We look for switching periods satisfying
\[
\frac{1}{2} \int_{kT}^{(k+1)T} \int_{\theta}^{(k+1)T} \| [\bar{A}(\theta), \bar{A}(\xi)] \|_P d\xi d\theta \exp \left( \int_{kT}^{(k+1)T} \mu_P(\bar{A}(\xi)) d\xi \right) + e^{T \mu_P(\bar{A})} < 1.
\]
By applying the triangle inequality and accounting for the boundedness of $B$, the double integral in (4.8) may be further bounded by
\[ \int_{kT}^{(k+1)T} \int_{0}^{(k+1)T} \|[A(\vartheta), A(\xi)]\|_P \, d\xi \, d\vartheta \leq \Gamma \rho_1 + \beta_1 \varepsilon + \beta_2 \varepsilon^2. \]

By applying property (2.3c) to the exponent in (4.4), by applying the norm triangle inequality to (4.7), and by using property (2.3d), the claim follows.

In the general case, when the state matrix is not of the form (4.6), the averaging method applies to systems of type (2.1) with state matrix $\varpi A(t)$, where $\varpi > 0$. The averaging method addresses the effects of $\varpi$ on the system stability. The small parameter $\varpi$ has the effect of rescaling the system (2.1) to the fast time $t/\varpi$. Indeed, by a change of variable, the system may be rewritten as
\[ \dot{x}(t) = A \left( \frac{t}{\varpi} \right) x(t). \]

Proposition 4.4. Under the hypotheses of Proposition 4.2 and assuming that $A$ is bounded, if there is a period $T$ such that
\[ \mu(\bar{A}_T(kT)) < -\beta \]
for all $k \in \mathbb{N}$ and for some positive $\beta$, then there exists an $\eta > 0$ such that the system (2.1) with state matrix $\varpi A(t)$, where $\varpi > 0$, is uniformly asymptotically stable for $\varpi T < \eta$.

Proof. From Proposition 4.2 the transition matrix of the system (2.1) with state matrix $\varpi A(t)$ from $kT$ to $(k+1)T$ is
\[ \Phi((k+1)T, kT) = \exp(\varpi \bar{A}_T(kT)T) + E_T(kT), \]
where $\|E_T(kT)\|$ is bounded according to (4.4). That is,
\[ \|E_T(kT)\| \leq \frac{1}{2} \varpi^2 \int_{kT}^{(k+1)T} \left\| [A(\vartheta), A(\xi)] \right\| d\sigma \, d\vartheta \times \exp \left( \varpi \int_{kT}^{(k+1)T} \mu(A(\xi)) \, d\sigma \right). \]

Since $A$ is bounded, the RHS of the previous inequality may be further bounded by
\[ g(\varpi T) = \frac{1}{2} \varpi^2 T^2 \varepsilon \exp(\varpi T \alpha), \]
where
\[ \alpha = \text{esssup}_{t \in \mathbb{R}^+} \mu(A(t)), \quad \varepsilon = \text{esssup}_{t, \xi \in \mathbb{R}^+} \| [A(t), A(\xi)] \|. \]
The function $g(\varpi T)$ is zero at $\varpi = 0$, as is its first derivative with respect to $\varpi$. By computing the norm of both sides of (4.10), by applying the norm triangle inequality and property (2.3d), and by accounting for (4.9), we obtain
\[ \|\Phi((k+1)T, kT)\| \leq \exp(-\varpi T \beta) + g(\varpi T). \]
The condition (4.9) is analogous to (3.3) in Proposition 3.1 and ensures that the matrix measure is uniformly negative for the more general system considered here. Thus by
selecting $\omega$ sufficiently small, $\Phi$ defines a contraction between any two consecutive instants $kT, (k+1)T$, and the system is asymptotically stable. $\square$

5. Conclusions. A novel framework for analyzing periodic linear switched systems has been presented. By combining results on exponential splitting methods with the method of averaging, we have provided an improved understanding of the switching mechanism. We have elucidated the importance of commutating relations among the different subsystems constituting the switched system on stability properties. A new estimate of the maximum switching period that guarantees that the switched system is uniformly asymptotically stable when the average system matrix is Hurwitz has been established. It has been shown that this estimate is less conservative than the standard estimate. Moreover, our estimate is consistent with sufficient stability conditions in the literature, such as in [4]. The estimate effectiveness has been validated through a sample problem from [19] and compared to other estimates and to the exact solution from Floquet theory. The effect of time-varying perturbations on the system stability has been discussed and a modified estimate of the switching rate accounting for additive perturbations has been developed. Moreover, the method has been generalized to arbitrary time-varying linear systems, and a sufficient condition for uniform asymptotic stability is stated in terms of some properties of sampled averages.

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REFERENCES

