[11] and [12] allow for the inclusion of a decay rate constraint, i.e., $\|e(t)\| < M e^{\alpha t}$. The inclusion of time-domain constraints to “shape” $e(t)$ is straightforward within the proposed approach.

Another difficulty with the $Q$-parametrization approach is the choice of the eigenvalues for the Ritz approximation. This is discussed in more detail in [26]. The $Q$-parametrization allows for an observer structure that includes state and estimator gain matrices [29]. This is required if the plant is not open-loop stable. However, the suboptimal state feedback controller from Section III could be used in the observer structure, and it is envisaged that this may improve convergence of the Ritz approximation. Determining an appropriate suboptimal estimator gain matrix remains for future work, as does consideration of the robust problem.

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Consensus Seeking Over Random Weighted Directed Graphs
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Abstract—We examine the consensus problem for a group of agents that communicate via a stochastic information network. Communication among agents is modeled as a weighted directed random graph that switches periodically. The existence of any edge is probabilistic and independent from the existence of any other edge. We further allow each edge to be weighted differently. Sufficient conditions for asymptotic almost sure consensus are presented for the case of positive weights and for the case of arbitrary weights.

Index Terms—Consensus problem, directed graphs, fast switching, random graphs, weighted graphs.

I. INTRODUCTION

In a consensus problem, a set of dynamic agents seeks to agree upon certain quantities of interests based upon shared information. Consensus problems are used to model many different phenomena involving information flow among agents, including flocking, swarming, synchronization, distributed decision making, and schooling; see, e.g., the survey paper [1].

Algebraic graph theory [2] is a natural framework for analyzing consensus problems; see, e.g., [3]–[6] and [7]. Within this framework, each agent is modelled as a vertex of a graph, and communication among Manuscript received January 24, 2006; revised August 28, 2006. Recommended by Associate Editor G. Pappas. This work was supported by the National Science Foundation under Grant IIS-0238092 and by the Office of Naval Research under Grants N000140510516, N000140310444, N000140510780, and N000140510516.

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two agents takes place when an edge interconnects the corresponding vertices of the graph.

Conditions for asymptotic consensus under a variety of assumptions on interagent communication have been recently reported by a number of investigators. Our contribution is to build upon and extend these recent contributions to include the case of stochastic directed graphs and arbitrarily weighted interagent communication. In [4], communication among agents is bidirectional and unity weighted, and the graph topology is determined randomly and independently of other edges with probability \( p_{ij} \in \{0, 1\} \). We collect the probabilities \( p_{ij} \) in the probability matrix \( P = [p_{ij}] \). We assume that the graph does not have self-loops. That is, no single edge starts and ends at the same vertex. In addition, we introduce the weight matrix \( W = [w_{ij}] \), where \( w_{ij} \neq 0 \) for \( i \neq j \) and \( w_{ii} = 0 \) for \( i \in V \). The adjacency matrix \( A = [a_{ij}] \) of a directed weighted random graph is a random matrix with all zeros on the main diagonal, and off-diagonal elements

\[
a_{ij} = \begin{cases} w_{ij} & \text{with probability } p_{ij} \\ 0 & \text{with probability } 1 - p_{ij}\end{cases} \tag{1}
\]

for \( i \neq j \). Similarly, the out-degree matrix \( D = \text{diag}[d] \) is a random matrix, whose nonzero elements are

\[
d_i = \sum_{j=1}^{n} a_{ij} \tag{2}
\]

The Laplacian matrix \( L = D - A \) is defined as the difference between the adjacency matrix \( A \) and the out-degree matrix \( D \). The finite sample space of the random graph is denoted \( \mathcal{G} \), the elementary events (possible graphs) are indicated by \( G^{(i)}, j = 1, \ldots, |\mathcal{G}| \), and \( |\mathcal{G}| \) indicates the cardinality of \( \mathcal{G} \). Connectedness of \( G \in \mathcal{G} \) plays an important role in our analysis. We say that a directed graph is strongly connected if there is a directed path from every node to every other node in the graph; see, for example, [5]. We define \( \pi = \Pr(G \in \mathcal{G} \text{ is strongly connected}) \), where \( \Pr \) indicates probability. The probability \( \pi \) indicates the probability that a graph \( G \in \mathcal{G} \) is strongly connected. Nonnegative matrices appear throughout the sequel. We say that a matrix \( B \) is nonnegative and write \( B \geq 0 \) if each entry of \( B \) is nonnegative. We say that \( B \geq C \) if \( B - C \) is a nonnegative matrix.

Remark 1: The Laplacian matrix \( L \) is a zero row-sum matrix. Therefore, the null space of \( L \) contains \( e = [1, \ldots, 1]^T \). In addition, \( G \) is strongly connected if and only if \( L \) is irreducible \([12, \text{Lemma 6.2.24}]\). Additional properties of directed weighted graphs may be found in the comprehensive paper \([13]\).

Since graph edges are independent random variables, the expected value of the graph Laplacian, say, \( E[L] = [\hat{l}_{ij}] \), may be defined entrywise by

\[
\hat{l}_{ij} = \begin{cases} -p_{ij}w_{ij} & i \neq j \\ \sum_{k=1}^{n} p_{ik}w_{ik} & i = j \end{cases} \tag{3}
\]

The matrix \( E[L] \) corresponds to a weighted directed graph that does not necessarily belong to \( \mathcal{G} \). We refer to this graph as the expected graph, denoted \( \hat{G} \).

Remark 2: If each edge has the same probability to exist, \( p_{ij} = p > 0 \) for \( i \neq j \) and \( i, j \in V \), the expected graph Laplacian \( E[L] \) is equal to the graph Laplacian of the complete graph multiplied by \( p \).

The complete graph is the strongly connected graph of \( G \), in which a weighted directed edge exists from each node to every other node.

B. Consensus Problem

We consider a dynamic system consisting of \( n \) agents interconnected pairwise via a unidirectional information network. Following \([3]\), we assume that the network is constant over each time interval \( \Delta > 0 \) and changes randomly at the transition instants \( t_k = k\Delta, \ k \in \mathbb{Z}^+ \), and that the communication pattern at the \( k \)th time-interval is completely unrelated to the communication pattern at the previous time-intervals.

The state of the agents at time \( t \) is described by the column vector \( x \), whose dynamics are \( \dot{x} = -Lx \). By introducing the sequence of random column vectors \( \{X_k = x(t_k)\} \), the consensus problem may be cast in a sample-data system setting, as done in \([3]\), by analyzing the stochastic linear system

\[
X_{k+1} = TX_k, \quad X_0 \in \mathbb{R}^n \tag{4}
\]
where $X_0$ indicates the initial state and $T$ is the random state matrix defined by

$$T = \exp(-\Delta L).$$  \hfill (5)

If all the elements of $X_k$ have the same value, then $X_k$ is in the span of $e = [1, \ldots, 1]^T$ and we say that the agents in the system are in agreement. We refer to the closed convex subset of $\mathbb{R}^n$ spanned by $e$ as the agreement space, denoted by $\mathcal{A}$. In addition to this sampled-data setting, the consensus problem can also be formulated in a discrete-time framework as done, for example, in [5], [8], and [14].

For any vector $x \in \mathbb{R}^n$, we define its disagreement $\delta$ with respect to some norm $\|\cdot\|$ as

$$\delta(x) = \min_{x^* \in \mathcal{A}} \|x - x^*\| = \|x - e\|$$  \hfill (6)

where the scalar $e$ is called the agreement of the vector $x$ and $e$ is the projection vector of $x$ onto $\mathcal{A}$. We associate the random sequence of disagreements $\{\delta_k\}$, defined by

$$\delta_k = \min_{x^* \in \mathcal{A}} \|X_k - x^*\|$$  \hfill (7)

with the random sequence of state vectors $\{X_k\}$. The random sequence $\{\delta_k\}$ represents the trajectory of the distance between the state vector and the agreement space.

**Definition 1:** We say that the set of agents whose evolution is determined by (4) and (5) asymptotically reaches consensus almost surely if $X_k \xrightarrow{a.s.} \mathcal{A}$.

This means that almost surely the agents will asymptotically agree on the same value. Equivalently, asymptotic consensus is achieved almost surely if the sequence of disagreements $\{\delta_k\}$ defined in (7) converges to zero almost surely

$$\delta_k \xrightarrow{a.s.} 0.$$  \hfill (8)

The definition of almost sure convergence may be found, for example, in [15, Ch. 5]. We note that consensus is not influenced by the choice of norm, since we are considering a finite number of agents, and all norms are equivalent in a finite-dimensional Banach space.

### III. Consensus Seeking: Positive Weights

In this section, we study the consensus problem stated in Definition 1 under the additional hypothesis that the edges’ weights are positive. That is, $w_{ij} > 0$ for $i \neq j$. From (1), this implies that, for any graph in $\mathcal{G}$, the corresponding adjacency matrix $A$ is nonnegative. Here we generalize the results of [3] to directed graphs where each link has a different probability to exist.

By adopting the infinity norm $\|\cdot\|_{\infty}$ and introducing the notation $\bar{x} = \max_{i=1,\ldots,n} x_i$ and $\underline{x} = \min_{i=1,\ldots,n} x_i$ for any vector $x \in \mathbb{R}^n$, the disagreement $\delta$ in (6) becomes

$$\delta(x) = \frac{\bar{x} - \underline{x}}{2}.$$  \hfill (9)

Thus, from Definition 1, the multiagent system asymptotically reaches consensus almost surely, if and only if the sequence of random disagreements $\{\delta_k\}$, with $\delta_k = (\bar{X}_k - \underline{X}_k)/2$, converges to zero almost surely.

The notions of pseudocontractive and nonexpansive matrices introduced in [10] are used extensively in the sequel and are recalled here for convenience.

**Definition 2:** We say that a matrix $B \in \mathbb{R}^{n \times n}$ is nonexpansive (with respect to $\|\cdot\|_{\infty}$ and $\mathcal{A}$) if

$$\|Bx - x^*\|_{\infty} \leq \|x - x^*\|_{\infty} \quad \text{for any } x \in \mathbb{R}^n, x^* \in \mathcal{A}$$  \hfill (10)

and pseudocontractive (with respect to $\|\cdot\|_{\infty}$ and $\mathcal{A}$) if, in addition to (10)

$$\delta(Bx) < \delta(x) \quad \text{for any } x \in \mathbb{R}^n, x \notin \mathcal{A}$$  \hfill (11)

holds.

**Remark 3:** For a nonexpansive matrix condition, (11) does not generally hold and the strict inequality should be relaxed to include equality.

### A. Analysis of the State Matrix: Basic Results

To develop general condition for asymptotic consensus that generalizes the result of [3] to positively weighted directed random graphs, we present several key properties of elementary events of the sample space $\mathcal{G}$. Throughout this analysis, we focus on a graph $G \in \mathcal{G}$ along with the corresponding graph Laplacian $L$ and state transition matrix $T$ defined in (5) with $\Delta > 0$.

**Lemma 1:** For a directed weighted graph $G$ with a corresponding adjacency matrix $A \geq 0$, $T$ is stochastic and has positive entries along the main diagonal for any $\Delta > 0$.

**Proof:** (Our argument is partially borrowed from the proof of [5, Lemma 3.11]). The Laplacian $L$ may be rewritten as $L = \Delta^{-1} - \bar{A}$, where $\bar{A}$ is the identity matrix in $\mathbb{R}^n$ and $\bar{A} = \max_{i=1,\ldots,n}(a_{ii})$. Note that $\bar{A}$ is nonnegative and

$$T = \exp(-\Delta L) \exp(\Delta \bar{A}) \geq \exp(-\Delta L)(I + \Delta \bar{A}).$$

Hence, $T$ is nonnegative and has positive entries on the main diagonal. In addition, since $\bar{e} = 0$ (see Remark 1), it follows that $Tx = e$, which means that $T$ is stochastic [12, Ch. 8.7].

**Lemma 2:** Suppose $G$ is a strongly connected directed weighted graph with corresponding adjacency matrix $A \geq 0$, and let $x, y \in \mathbb{R}^n$ be such that $\bar{x} \geq \underline{x}$, $y = Tx$, and $\Delta > 0$. Then $\bar{x} \leq y \leq \bar{x}$, and the number of the elements in the set $\{i : y_i = \bar{x} \text{ or } y_i = \underline{x}\}$ is at least one less than the number of elements in $\{i : x_i = \bar{x} \text{ or } x_i = \underline{x}\}$.

**Proof:** From Lemma 1, $T$ is stochastic and has positive entries along the main diagonal. In addition, from Remark 1, we know that $L$ is irreducible since $G$ is strongly connected. Since $T$ and $L$ share the same invariant spaces, it follows that $T$ is also irreducible. Therefore, by applying [10, Proposition 3.2], the claim follows.

**Lemma 3:** For a directed weighted graph $G$ with a corresponding adjacency matrix $A \geq 0$, $T$ is nonexpansive. If, in addition, $G$ is strongly connected, $T$ is pseudocontractive.

**Proof:** Since $T$ is stochastic, $\|T\|_{\infty} \leq 1$, from the definition of the matrix infinity norm. Also, $Tx^* = x^*$ is satisfied for all $x^* \in \mathcal{A}$. These two facts yield

$$\|Tx - x^*\|_{\infty} = \|T(x - x^*)\|_{\infty} \leq \|x - x^*\|_{\infty}$$

for any $x \in \mathbb{R}^n$, $x^* \in \mathcal{A}$, which by Definition 2 completes the first part of claim. The arguments of the proof of the second part of the lemma are partially borrowed from the proof of [7, Lemma 1]. Consider the switching period $\Delta_t = \Delta/(n-1)$ and an arbitrary vector $x \notin \mathcal{A}$, which has its largest entry different from the smallest one. Define the transition matrix $T_{\Delta_t} = \exp(-\Delta \bar{A})L$ and note that $T_{\Delta_t}^{n-1} = T$. Applying Lemma $2n-1$ times to $T_{\Delta_t}$, at least one of the sets $\{i : y_i = \bar{x} \}$ and $\{i : y_i = \underline{x}\}$ is empty, where $y = Tx$. We address the case that $i : y_i = \bar{x}$ is empty, with understanding that the other case that $i : y_i = \underline{x}$ is empty proceeds in parallel manner. Suppose $i : y_i = \bar{x}$ is empty, and therefore $y_i > \underline{x}$. From (9), we obtain

$$\delta(y) = \frac{\bar{y} - \underline{y}}{2} \leq \frac{\bar{x} - \underline{x}}{2} < \frac{\bar{x} - \underline{x}}{2} = \delta(x) z$$

which by Definition 2 yields the second part of the claim.
The concept of pseudocontractivity was first introduced in [10] for analyzing nonstationary iterative methods for linear systems. A similar concept may be found in the distributed computation algorithm proposed in [16].

B. Ascertain Consensus

Here we apply the results of Section III-A to the analysis of the consensus problem posed in Definition 1 for positively weighted random graphs.

Lemma 4: For $\lambda > 0$, define $Q_{\lambda} = \{ x \in \mathbb{R}^n : \delta(x) < \lambda \}$. Suppose the sequence $\{X_k\}$ satisfies the dynamic system (4) and all nondiagonal entries of the weight matrix $W^\ast = [w_{ij}]$ are positive. If $X_0 \in Q_{\lambda}$, then $X_k \in Q_{\lambda}$ for all $k \in \mathbb{Z}^+$ with probability one. Equivalently

$$\Pr\left( \sup_{k \geq 0} \delta(X_k) \geq \lambda \right) = 0.$$

Proof: From Lemma 3 and Remark 3, for any $k \in \mathbb{Z}^+$, $\delta(X_{k+1}) \leq \delta(X_k)$, which proves the claim. \hfill $\blacksquare$

Proposition 1: Consider the stochastic dynamic system in (4). If $\pi = \Pr\{G \in G_0 \text{ is strongly connected} \}$ is nonzero, and all nondiagonal entries of the weight matrix are positive, then the multiagent system asymptotically reaches consensus almost surely.

Proof: Consider the quantity

$$g(x) = \delta(x) - E[\delta(X_{k+1}) | X_k = x]$$

on the set $Q_{\lambda} = \{ x \in \mathbb{R}^n : \delta(x) < \lambda \}$, where $E[\delta(X_{k+1}) | X_k = x]$ is the conditional expected value of $\delta(X_{k+1})$ given $X_k = x$. From Lemma 4, if $X_0 \in Q_{\lambda}$, then $X_k \in Q_{\lambda}$ with probability one. Our strategy is to show that $g(X_k)$ converges almost surely to the origin, which implies that the multiagent system asymptotically reaches consensus almost surely. The function $g$ vanishes on $\mathcal{A}(Q_{\lambda} \setminus \mathcal{A})$ since $\delta(x)$ and $\delta(T(x))$ both vanish. In addition, the function $g$ is strictly positive on $\{ x \in Q_{\lambda} : x \notin \mathcal{A} \}$. Indeed, consider

$$E[\delta(X_{k+1}) | X_k = x] = \sum_{j=1}^{[g]} \delta\left(T^{(j)}x\right)p^{(j)}$$

where $T^{(j)}$ is the state matrix corresponding to the graph $G^{(j)}$ in $\mathcal{C}$, and $p^{(j)}$ indicates its probability. Since $\pi > 0$, at least one $G^{(j)}$ from $\mathcal{C}$ is strongly connected, and corresponding probability $p^{(j)}$ is nonzero. Thus, from Lemma 3, the right-hand side of (12) is strictly less than $\delta(x)$ for $x \notin \mathcal{A}$. Therefore, from [17, Th. 1], $g(X_k) \xrightarrow{a.s.} 0$ for all the random sequences with $X_0 \in Q_{\lambda}$. Since $g$ is continuous, and is zero only on the agreement space $\mathcal{A}$, the multiagent system asymptotically reaches consensus almost surely. \hfill $\blacksquare$

The mathematical ingredients used in the proof of Proposition 1 are similar to those used in [14], with the main difference that here we consider a stochastic framework for the consensus problem. The following results provide sufficient conditions for consensus in terms of the expected graph Laplacian.

Lemma 5: The probability $\pi$ is nonzero if and only if the expected graph $\bar{G}$, whose Laplacian matrix is defined in (3), is strongly connected.

Proof: If $\bar{G}$ is strongly connected, then there is a strongly connected graph $G \in \mathcal{C}$ whose probability is nonzero. The graph $\bar{G}$ is obtained by connecting the vertices through the edges that have nonzero probability. On the other hand, if $\pi$ is nonzero, then there exists at least one strongly connected graph in $\mathcal{C}$ whose probability is nonzero. Since the set of edges in $G$ with nonzero probability are contained in set of edges in $\bar{G}$, it must be that $\bar{G}$ is strongly connected. \hfill $\blacksquare$

Proposition 1 may be restated in light of the conclusion of Lemma 5.

Proposition 2: Consider the stochastic dynamic system in (4). If the expected graph $\bar{G}$ is strongly connected and all nondiagonal entries of the weight matrix are positive, then the system asymptotically reaches consensus almost surely.

The following is an immediate consequence of Proposition 1 and Lemma 5, and it is a generalization of the results in [3] for directed, positively weighted graphs.

Corollary 1: Consider the stochastic dynamic system in (4). If all nondiagonal entries of the weight matrix are positive and $p_{ij} = p > 0$, the multiagent system reaches asymptotically consensus almost surely.

IV. CONSENSUS SEEKING: ARBITRARY WEIGHTS

We now consider the case of arbitrary weights that are not necessarily positive. Connectedness of the expected graph $\bar{G}$ is generally not sufficient for the agents to asymptotically reach consensus when arbitrary nonzero weights are assumed. Indeed, the state matrix $T$ of a graph with arbitrary nonzero weights may not be nonexpansive.

We introduce the matrix $Q \in \mathbb{R}^{n \times n}$ such that $Q^T e = 0$ and $Q^T Q = I_{n-1}$, and we define the random sequence $\{\xi_k = Q^T X_k\}$. Note that $Q$ is an isomorphism between $\mathbb{R}^n$ and the hyperplane $\mathbb{R}^{n-1}$. The vector $X_k$ may be decomposed as $X_k = Q\xi_k + (1/n)(e^T X_k)e$, and the stochastic (4) can be expressed in terms of the random sequence $\{\xi_k\}$ as

$$\xi_{k+1} = \bar{T}\xi_k$$

where

$$\bar{T} = Q^T T Q.$$

Adopting the Euclidean norm, the disagreement vector defined in (6) of any vector $x \in \mathbb{R}^n$ can be expressed as $\|x\|_2 = \|Q^T x\|_2$. By Definition 1, this system asymptotically reaches consensus almost surely if the sequence $\{\delta_k = \|Q\xi_k\|_2\}$ converges to zero almost surely. This is equivalent to requiring that the sequence $\{\|\xi_k\|\}$ converges to zero almost surely, where $\|\cdot\|$ is a norm in $\mathbb{R}^{n-1}$.

Proposition 3: Consider the stochastic dynamic system in (4). If the spectrum of $E[L]$ has $n-1$ eigenvalues in the open right-half complex plane, then there exists a switching period $\Delta^*$ such that when $\Delta < \Delta^*$, the multiagent system asymptotically reaches consensus almost surely.

Proof: From [17, Th. 8], if there exist positive-definite matrices $B$ and $C$ such that

$$E[\bar{T}^T B \bar{T}] - B = -C$$

then $\xi_{k+1} C \xi_{k+1} \xrightarrow{a.s.} 0$. By defining the norm $\|\xi\|^2 = \xi^T C \xi$, it is clear that $\|\xi_k\| \xrightarrow{a.s.} 0$ and consensus is asymptotically achieved almost surely. The expected value of the graph Laplacian $E[L]$ has always $e \in \mathbb{R}$ in its null space, since it is a linear combination of graph Laplacians. If the remaining $n-1$ eigenvalues of $E[L]$ are in the open right-half complex plane, the matrix $-E[L] = -Q^T E[L] Q$ is Hurwitz. Thus, we choose $B$ in (15) as the unique solution of

$$E[L]^T B + B E[L] = I_{n-1}.$$

The series expansion of $\bar{T}$ in (14) at $\Delta = 0$ is

$$\bar{T} = I_{n-1} - \Delta \bar{L} + O(\Delta^2).$$
By replacing the expression for \( \tilde{T} \) given in (17) in the left-hand side (LHS) of (15) and by accounting for (16), we obtain that the LHS of (15) equals

\[-\Delta T_{n-1} + O(\Delta^2)\]  

(18)

which is negative-definite for sufficiently small \( \Delta > 0 \).

Proposition 3 is directly applicable to the positively weighted directed graphs addressed in Section III. Indeed, the Laplacian of a positively weighted, directed, strongly connected graph has \( n-1 \) eigenvalues in the open right-half complex plane (see, e.g., [6]). Nevertheless, this yields a much weaker result than those implied by Proposition 2, since it only states that consensus may be asymptotically achieved almost surely if the topology changes sufficiently fast. The analysis presented in Section III shows that, for positively weighted, directed graphs, consensus is achieved regardless of the switching period.

By applying Theorem 1 in [18], the critical switching rate \( \Delta^* \) may be found by looking for the smallest value of \( \Delta \) that forces the eigenvalues of the matrix \( E[T^t \otimes \tilde{T}^t] \) to leave the open unit circle. For a relatively large graph, the analysis of the spectrum of \( E[T^t \otimes \tilde{T}^t] \) is computationally expensive. Conservative estimates may be derived by applying fast switching results similar to those presented for deterministic systems in [19], [9], and [20].

V. SIMULATIONS RESULTS

To illustrate our principal conclusions, we present the results of numerical simulations consisting of ten agents and a random weighted directed network. The initial condition \( X_0 \) is a random vector whose entries are uniformly distributed between zero and one. The disagreement is computed using the \( \infty \)-norm as in (9).

Figs. 1 and 2 depict the disagreement versus the time-step \( k \) for a positively weighted directed random graph with \( \Delta = 0.5 \).
In Fig. 1 the graph is unity weighted, and each edge is assigned the same probability to exist, $p_{ij} = p = 0.01$. In Fig. 2, the graph is weighted in such a way that $W = \text{circ}\{0.2, 2, 2, 1, 1, 1, 1, 1\}$ and the edge probabilities are chosen in such a way that $P = \text{circ}\{0.04, 0.02, 0.01, 0.005, 0.005, 0.005, 0.005, 0.005, 0.002, 0.004\}$. We indicate with $\text{circ}\{r\}$ the circulant matrix whose rows are composed of cyclically shifted versions of the list $r$. In both cases, the probability that the graph is strongly connected is nonzero, and the agents asymptotically reach consensus illustrating the claim of Proposition 2.

Fig. 3 shows the disagreement $\delta$ versus the time-step $k$ for an arbitrarily weighted directed random graph with $\Delta = 0.05, W = \text{circ}\{0.2, 2, 2, 1, 1, 1, -1, -1\}$ and $P = \text{circ}\{0.04, 0.02, 0.01, 0.005, 0.005, 0.005, 0.005, 0.005, 0.02, 0.04\}$. Although not shown, if $\Delta = 0.5$, consensus is not achieved. Fig. 4 shows the results of a Monte Carlo simulation over a population of 100 simulation runs for the case considered by Figs. 3. Figs. 3 and 4 illustrate the claim of Proposition 3. Consensus may be achieved through fast switching if the graph Laplacian of the expected graph has $n-1$ eigenvalues in the open right-half half-plane.

V. CONCLUSION

New generalizations on consensus seeking for arbitrary weighted directed random graphs are presented. We show that, for positively weighted graphs, consensus is asymptotically achieved if the probability that the random network is strongly connected is nonzero. Further, we define an expected graph and show that the probability that the random network is strongly connected is nonzero if and only if the expected graph is strongly connected. For the case that weights are arbitrary, we utilize tools based on fast switching and perturbation methods and show that consensus may be asymptotically achieved if
certain spectral properties of the average graph Laplacian are satisfied and if the switching period is sufficiently small.

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H∞ Control and Estimation of Retarded State-Multiplicative Stochastic Systems

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Abstract—Linear, state-delayed, continuous-time systems with stochastic uncertainties in their state-space model are considered. The problems of both H∞ state-feedback control and filtering are solved, for the stationary case, via an input–output approach by which the system is replaced by a nonretarded system with deterministic norm-bounded uncertainties. In both problems, a cost function is defined that is the expected value of the standard H∞ performance index with respect to the uncertain parameters. Three examples are given that demonstrate the applicability of the theory.

Index Terms—Stochastic H∞ control, time-delay systems, uncertain systems.

I. INTRODUCTION

In this paper, we address the problems of H∞ state-feedback control and estimation of state-delayed, continuous-time, state-multiplicative linear systems via the input–output approach based on [2, Lemma 1]. The multiplicative noise appears in both the delayed and the nondelayed states of the system.

The analysis and design of controllers for systems with stochastic uncertainties have received much attention in the past (see [3] and references therein) where mainly the stability issue and state-feedback control problems have been considered. Recently, a renewed interest in these problems and related ones has been encountered, and solutions to the stochastic control and filtering problems have been derived that ensure a worst case performance bound in the H∞ style [3]–[12].

Systems whose parameter uncertainties are modeled as white noise processes in a linear setting have been treated in [5]–[9] for the continuous-time case and in [4] and [10]–[12] for the discrete-time case. Such models of uncertainties are encountered in many areas of applications (see [4] and references therein) such as nuclear fission and heat transfer, population models, and immunology. In control theory, such models are encountered in gain scheduling when the scheduling parameters are corrupted with measurement noise. Most practically, in [12], a discrete-time stochastic estimation for a guidance-motivated tracking problem was solved and was shown to achieve better results than those achieved by the Kalman filter.

The stability and control of deterministic delayed systems of various types (i.e., constant time delay, slow and fast varying delay, etc.) has been a central field within the system theory sciences. In the last two decades, systems with uncertain time-delay have been a subject of recurring interest, especially due to the emergence of the H∞ control theory in the early 1980s. Most of the works are based on application of different types of Lyapunov Krasovskii functionals (LKFs) (see, for example, [13]–[17]). Also continuous-time systems with time-varying delays, which are confined by $\frac{dh}{dt} < 1$, where $h$ is the delay interval, were treated via descriptor-type LKF [17], where the derivative of the
delayed systems.

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